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Group Invariance Properties of the Poisson-Boltzmann and Other Nonlinear Field Equations

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by

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by

Roy Arthur Axford

ABSTRACT

Group invariance properties of ordinary, nonlinear differential equations that occur in the elementary theory of the thermal and submarine explosion problems are established and applied to the development of further analytic solutions of these differential equations.

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1. INTRODUCTION

The fact that certain nonlinear field equations, which occur in the elementary analysis of the thermal explosion and submarine explosion problems, admit various finite, continuous groups of point transformations apparently has not been recognized previously. The precise nature of this fact is established in this report and is applied to obtain both a group theoretic interpretation of the integrability and further integrals of the nonlinear field equations that arise in these problems.

2. GROUP INVARIANCE PROPERTIES OF THE POISSON-BOLTZMANN EQUATION

2.1 Introduction

In this section we establish certain group invariance properties that pertain to the Poisson-Boltzmann equation, namely,

$$\nabla^2 y + s\, e^y = 0, \; (s > 0), \qquad (2\text{-}1)$$

in one-dimensional plane, cylindrical, and spherical geometries. The fact that the Poisson-Boltzmann equation admits various two-parameter groups is applied to the problem of obtaining general integrals of this equation in explicit form for plane and infinite cylindrical geometries. The absence of a two-parameter group under which Eq. (2-1) is invariant, as in spherical symmetric geometry, provides an interpretation that Eq. (2-1) cannot be integrated by quadratures in this geometry, although it does admit a one-parameter group in this geometry.

2.2 Invariance under the Translation Group in Plane Geometry

In plane geometry, Eq. (2-1) becomes

$$y'' + s\, e^y = 0. \qquad (2\text{-}2)$$

This differential equation is invariant under the one-parameter group of translations parallel to the x-axis of the x-y-plane because the independent variable does not appear explicitly. Any autonomous differential equation in two variables is invariant under this translation group irrespective of the order of the differential equation.

A second-order, ordinary differential equation that is known to be invariant under a transformation group can be reduced to one of first order. This can be accomplished by the introduction of a first differential invariant of the group as a new dependent variable, and of an invariant of the group as a new independent variable. The determination of an invariant and a first differential invariant of the group requires the calculation of two linearly independent solutions of the linear, first-order, partial differential equation obtained from the symbol of the first extension of the infinitesimal transformation of the group. The group of translations under which Eq. (2-2) is invariant can be generated from the infinitesimal transformation represented by the symbol

$$Uf = \frac{\partial f}{\partial x} . \qquad (2\text{-}3)$$

Because this is also the symbol of the once-extended group, an invariant and first differential invariant are solutions of

$$U'f(x,y,y') \equiv \frac{\partial}{\partial x} f(x,y,y') = 0, \qquad (2\text{-}4)$$

with the equivalent first-order system

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dy'}{1} . \qquad (2\text{-}5)$$

Accordingly,

$$u(x,y) = y \qquad (2\text{-}6)$$

is an invariant, and

$$u'(x,y,y') = y' \qquad (2\text{-}7)$$

is a first differential invariant of the group of translations. Upon introducing the new variables defined by

$$Y = u'(x,y,y') = y' \qquad (2\text{-}8)$$

and

$$X = u(x,y) = y \qquad (2\text{-}9)$$

into Eq. (2-2), we obtain

$$Y\,dY + s\,e^X\,dX = 0, \qquad (2\text{-}10)$$

because

$$\frac{dY}{dX} = \frac{\frac{\partial Y}{\partial x}dx + \frac{\partial Y}{\partial y}dy + \frac{\partial Y}{\partial y'}dy'}{\frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial y}dy} = \frac{y''}{y'} , \qquad (2\text{-}11)$$

so that

$$y'' = y'\frac{dY}{dX} = Y\frac{dY}{dX} . \qquad (2\text{-}12)$$

Integration of Eq. (2-10) produces

$$Y^2 = C_1 - s\,2\,e^X , \qquad (2\text{-}13)$$

in which C_1 is an arbitrary constant. With Eqs. (2-8) and (2-9), this last relation becomes

$$\frac{dy}{dx} = \sqrt{C_1 - 2\,s\,e^y} , \qquad (2\text{-}14)$$

so that

$$x = C_2 + \int dy[C_1 - 2\,s\,e^y]^{-1/2} , \qquad (2\text{-}15)$$

where C_2 is the second arbitrary constant, is the general solution of Eq. (2-2).

The above procedure, used to find the general solution of Eq. (2-2) as given by Eq. (2-15), comprises the group theoretic interpretation of the elementary integration procedure described as follows. If the independent variable is missing in a second-order, ordinary differential equation, let $y' = p$ and $y'' = p\,dp/dy$, determine the function $p = p(y)$, and then integrate the resulting separable differential equation $y' = p$. However, the group theoretic method can be applied to any second-order differential equation,

$$F(x,y,y',y'') = 0, \qquad (2\text{-}16)$$

that is known to be invariant under a one-parameter group of point transformations even if the independent variable appears explicitly. The elementary procedure may not be so applied because it depends upon the absence of the independent variable.

2.3 A Second One-Parameter Group Admitted by $y'' + s\exp(y) = 0$

The invariance of Eq. (2-2) under the translation group is apparent because of the absence of the independent variable. A second-order, ordinary differential equation is, at most, invariant under eight linearly independent, one-parameter groups, although it need not necessarily admit any group. The question then arises of finding any further groups admitted by a second-order differential equation, if in fact one is admitted. We now prove the following proposition.

Proposition 2.1. The second-order differential equation, $y'' + s\exp(y) = 0$, is invariant under the one-parameter group of point transformations that is generated by the infinitesimal transformation with the symbol

$$Uf = x\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} . \qquad (2\text{-}17)$$

Proof: The symbol of the once-extended group generated by Eq. (2-17) is

$$U'f = x\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} - y'\frac{\partial f}{\partial y'} , \qquad (2\text{-}18)$$

and the first-order partial differential equation that corresponds to $y'' + s\exp(y) = 0$ is

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} - s\exp(y)\frac{\partial f}{\partial y'} = 0. \qquad (2\text{-}19)$$

The commutator constructed from the operators that appear in Eqs. (2-18) and (2-19) assumes the value

$$(U'A)f = -\frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y} + s\exp(y)\frac{\partial f}{\partial y'} = -Af. \qquad (2\text{-}20)$$

Therefore, Eq. (2-19) is invariant under the group generated by Eq. (2-17) because the invariance condition, $(U'A)f = \lambda(x,y,y')\,Af$, is satisfied with $\lambda = -1$. Accordingly, the second-order differential equation in question also admits the group generated by Eq. (2-17) because it is equivalent to the partial differential equation in Eq. (2-19).

Although $y'' + s\exp(y) = 0$ is invariant under the one-parameter group generated by Eq. (2-17), it is not the most general form of a second-order, ordinary differential equation that admits this group. This form is contained in the following result.

Proposition 2.2. The general form of a second-order, ordinary differential equation that admits the one-parameter group generated by the infinitesimal transformation,

$$Uf = x\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y}, \qquad (2\text{-}21)$$

is contained in the relation,

$$f(x^2y'', xy', y + \ln x^2) = 0, \qquad (2\text{-}22)$$

in which f is an arbitrary function of the three indicated arguments.

Proof: The symbol of the second extension of the group generated by Eq. (2-21) is

$$U''f = x\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} - y'\frac{\partial f}{\partial y'} - 2y''\frac{\partial f}{\partial y''}. \qquad (2\text{-}23)$$

The first-order system that corresponds to $U''f = 0$ is

$$\frac{dx}{x} = -\frac{dy}{2} = -\frac{dy'}{y'} = -\frac{dy''}{2y''}. \qquad (2\text{-}24)$$

Accordingly, an invariant of the group under question comes out of

$$\frac{dx}{x} = -\frac{dy}{2} \qquad (2\text{-}25)$$

in the form

$$u(x,y) = y + \ln x^2; \qquad (2\text{-}26)$$

a first differential invariant out of

$$\frac{dx}{x} = -\frac{dy'}{y'} \qquad (2\text{-}27)$$

in the form

$$u'(x,y,y') = xy'; \qquad (2\text{-}28)$$

and a second differential invariant out of

$$\frac{dx}{x} = -\frac{dy''}{2y''} \qquad (2\text{-}29)$$

in the form

$$u''(x,y,y',y'') = x^2y''. \qquad (2\text{-}30)$$

The arbitrary function of this invariant and first and second differential invariants given in Eq. (2-22) is, then, the general form of a second-order, ordinary differential equation that admits the one-parameter group of point transformations generated by the infinitesimal transformation with the symbol of Eq. (2-21).

2.4 A Two-Parameter Group Admitted by $y'' + s\exp(y) = 0$

A second-order, ordinary differential equation that admits a two-parameter group of point transformations is integrable by quadratures. Accordingly, the following result is of interest.

Proposition 2.3. The second-order differential equation, $y'' + s\exp(y) = 0$, is invariant under the two-parameter group of point transformations generated by the two basis transformations,

$$U_1f = \frac{\partial f}{\partial x}, \qquad (2\text{-}31)$$

and

$$U_2f = x\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y}. \qquad (2\text{-}32)$$

Proof: The invariance of the differential equation under each of these transformations has already been shown. Therefore, it is sufficient to show that U_1f and U_2f comprise the basis of a two-parameter group, which will be the case if their commutator is a linear combination of the basis transformations, that is, if

$$(U_1U_2)f = e_1U_1f + e_2U_2f \qquad (2\text{-}33)$$

in which e_1 and e_2 are constants. The proposition is established by evaluating the commutator:

$$(U_1U_2)f = U_1f. \qquad (2\text{-}34)$$

The two-parameter group generated by the infinitesimal transformations that appear in Eqs. (2-31) and (2-32) is of the third type in the sense of Lie's definition of the four basic types of two-parameter groups. The most general form of second-order, ordinary differential equation, which is invariant under this two-parameter group, is not $y'' + s\exp(y) = 0$, but that contained in the following result.

Proposition 2.4. The general form of a second-order, ordinary differential equation that is invariant under the two-parameter group generated by the basis,

$$U_1f = \frac{\partial f}{\partial x}, \qquad (2\text{-}35)$$

and

$$U_2f = x\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} \qquad (2\text{-}36)$$

is given by the relation

$$f\left[\frac{y''}{(y')^2}, y - 2\ln y'\right] = 0, \qquad (2\text{-}37)$$

in which f is an arbitrary function of the two indicated arguments.

Proof: The second extensions of the two basis transformations given in Eqs. (2-35) and (2-36) are

$$U_1''f = \frac{\partial f}{\partial x}, \qquad (2\text{-}38)$$

and

$$U_2''f = x\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} - y'\frac{\partial f}{\partial y'} - 2y''\frac{\partial f}{\partial y''}. \qquad (2\text{-}39)$$

Each of these last two operators annihilates the relation of Eq. (2-37), a fact that establishes the proposition.

A further result relative to the group invariance properties of $y'' = s\exp(y) = 0$ follows.

Proposition 2.5. The second-order differential, $y'' + s\exp(y) = 0$, is invariant under r-parameter groups of point transformations for the cases of $r = 1$ and $r = 2$, but not for the case of $r \geq 3$. The proof that establishes this result is too space-consuming to be included here.

2.5 Integration of $y'' + s\exp(y) = 0$ by the Utilization of the Two-Parameter Group It Admits

The invariance result stated in Proposition 2.3 may be exploited to obtain the general solution of $y'' + s\exp(y) = 0$ in analytic form. The canonical variables of the two-parameter group generated by the basis of Eqs. (2-35) and (2-36) are

$$X = \exp(-y/2) \qquad (2\text{-}40)$$

and

$$Y = x + \exp(-y/2). \qquad (2\text{-}41)$$

Introduction of these new variables into $y'' + s\exp(y) = 0$ produces

$$X\frac{d^2Y}{dX^2} = \left(1 - \frac{dY}{dX}\right)\left[1 + \frac{s}{2}\left(1 - \frac{dY}{dX}\right)^2\right]. \qquad (2\text{-}42)$$

The general solution of this equation is

$$Y = X \pm \sqrt{\frac{2C_1}{s}}\ln\left[X + \sqrt{X^2 - C_1}\right] + C_2, \qquad (2\text{-}43)$$

in which C_1 and C_2 are arbitrary constants, and the choice of the sign depends upon the boundary conditions.

By reverting to the original variables, we find that

$$x = C_2 \pm \sqrt{\frac{2C_1}{s}}\ln\left[\exp(-y/2) + \sqrt{\exp(-y) - C_1}\right], (2\text{-}44)$$

the general solution of $y'' + s\exp(y) = 0$.

If this solution is now subjected to the boundary conditions, $y'(0) = 0$ and $y(1) = 0$, we find that the positive sign in Eq. (2-43) is to be taken and that

$$C_1 = \exp(-y_o), \qquad (2\text{-}45)$$

where y_o is the value of the solution at $x = 0$, and

$$C_2 = 1 - \sqrt{\frac{2}{s}\exp(-y_o)}\ln\left[1 + \sqrt{1 - \exp(-y_o)}\right]. \qquad (2\text{-}46)$$

It follows that

$$x = 1 + \sqrt{\frac{2}{s}\exp(-y_o)}\ \ln\left[\frac{\exp(-y/2) + \sqrt{\exp(-y) - \exp(-y_o)}}{1 + \sqrt{1 - \exp(-y_o)}}\right] \qquad (2\text{-}47)$$

for the above boundary conditions. Resolving this last relation with respect to y produces

$$y = 2\ln\cosh\sqrt{\frac{s}{2}\exp(y_o)} - 2\ln\cosh\left(x\sqrt{\frac{s}{2}\exp(y_o)}\right). \qquad (2\text{-}48)$$

The value of the solution at $x = 0$ is a root of the transcendental relation,

$$\exp(y_o/2) = \cosh\left(\sqrt{\frac{s}{2}\exp(y_o)}\right), \qquad (2\text{-}49)$$

so that Eq. (2-48) may also be written as

$$y = y_o - 2\ln\cosh\left(x\sqrt{\frac{s}{2}\exp(y_o)}\right). \qquad (2\text{-}50)$$

This last relation is the solution of $y'' + s\exp(y) = 0$ subject to the boundary conditions, $y'(0) = 0$ and $y(1) = 0$, provided that a root of Eq. (2-49) exists. There are two roots of Eq. (2-49) if $0 < s < 0.88$, and none if $s > 0.88$.[1] The result contained in Eq. (2-50) agrees with that given by Carslaw and Jaeger,[1] who obtained it by a different method. The general solution of $y'' + s\exp(y) = 0$ contained in Eq. (2-44), and obtained by methods based upon invariance properties, leads to Eq. (2-50) as a special case for particular boundary conditions.

2.6 A One-Parameter Group under Which $\nabla^2 y + s\exp(y) = 0$ is Invariant in One-Dimensional Plane, Cylindrical, or Spherical Geometry

The differential equation, $\nabla^2 y + s\exp(y) = 0$, possesses the rather remarkable property of being invariant under the same group of point transformations in one-dimensional plane, cylindrical, and spherical geometries. This property is embodied in the following result.

Proposition 2.6. The second-order, ordinary differential equation,

$$x^2y'' + N\,xy' + s\exp(y + \ln x^2) = 0, \qquad (2\text{-}51)$$

is invariant under the one-parameter group of point transformations generated by the infinitesimal transformation,

$$Uf = x \frac{\partial f}{\partial x} - 2 \frac{\partial f}{\partial y} , \qquad (2\text{-}52)$$

for all values of the constant, N. If $N = 0$, then Eq. (2-51) is $\nabla^2 y + s \exp(y) = 0$ for plane geometry. The case of $N = 1$ corresponds to infinite cylindrical geometry, and $N = 2$ is the case of spherical geometry with spherical symmetry.

Proof: This proposition is a direct consequence of Proposition 2.2 when the arbitrary function indicated in Eq. (2-22) is taken so as to give Eq. (2-51).

Section 2.4 shows that $\nabla^2 y + s \exp(y) = 0$ is invariant under a two-parameter group in the case of plane geometry. In spherical geometry with spherical symmetry this differential equation admits only a single one-parameter group.

Proposition 2.7. The second-order, ordinary differential equation,

$$x^2 y'' + 2 xy' + s x^2 \exp(y) = 0, \qquad (2\text{-}53)$$

is invariant only under the one-parameter group generated by the infinitesimal transformation

$$Uf = x \frac{\partial f}{\partial x} - 2 \frac{\partial f}{\partial y} . \qquad (2\text{-}54)$$

Because of its length, the proof of this result will be omitted here.

A practical consequence of Proposition 2.7 is that it provides a group theory argument for the fact that we should not expect the Poisson-Boltzmann equation in spherically symmetric geometry to be integrable by quadratures alone.

In the case of infinite cylindrical geometry, Eq. (2-51) admits further groups beyond that generated by the symbol of Eq. (2-52).

2.7 Further Groups Admitted by $\nabla^2 y + s \exp(y) = 0$ in Infinite Cylindrical Geometry

An additional one-parameter group and a two-parameter group of point transformations are admitted by $\nabla^2 y + s \exp(y) = 0$ in the case of infinite cylindrical geometry.

Proposition 2.8. The nonlinear, second-order differential equation,

$$x^2 y'' + xy' + s x^2 \exp(y) = 0, \qquad (2\text{-}55)$$

is invariant under the one-parameter group of point transformations generated by the infinitesimal transformation represented by the symbol

$$Uf = x \ln x \frac{\partial f}{\partial x} - 2 (1 + \ln x) \frac{\partial f}{\partial y} . \qquad (2\text{-}56)$$

Proof: The symbol of the once-extended group represented by Eq. (2-56) is

$$U'f = x \ln x \frac{\partial f}{\partial x} - 2 (1 + \ln x) \frac{\partial f}{\partial y} - \left[\frac{2}{x} + (1 + \ln x) y'\right] \frac{\partial f}{\partial y'} , \qquad (2\text{-}57)$$

and the linear, first-order partial differential equation that is equivalent to Eq. (2-55) is

$$Af \equiv \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} - \left[\frac{y'}{x} + s \exp(y)\right] \frac{\partial f}{\partial y'} = 0. \qquad (2\text{-}58)$$

Because the commutator that comes out of the operators appearing in Eqs. (2-57) and (2-58) assumes the value

$$(U'A)f = - (1 + \ln x) Af, \qquad (2\text{-}59)$$

the proposition is established because the invariance condition, $(U'A)f = \lambda(x,y,y')Af$, is satisfied with $\lambda = - (1 + \ln x)$.

The invariance property stated in Proposition 2.8 is a special case of the result that follows.

Proposition 2.9. The second-order, ordinary differential equation contained in the relation, in which f denotes an arbitrary function of the three given arguments,

$$f\left[y + 2 \ln (x \ln x), (xy' + 2) \ln x, (x^2 y'' + xy')(\ln x)^2\right] = 0, \qquad (2\text{-}60)$$

is the general form of such an equation that admits the one-parameter group of point transformations with the infinitesimal transformation,

$$Uf = x \ln x \frac{\partial f}{\partial x} - 2 (1 + \ln x) \frac{\partial f}{\partial y} . \qquad (2\text{-}61)$$

Proof: This proposition can be established by determining an invariant and first and second differential invariants of the group in question by computing three functionally independent integrals of the linear, first-order partial differential equation obtained with the symbol of the second extension of the group. This partial differential equation is

$$U''f \equiv x \ln x \frac{\partial f}{\partial x} - 2 (1 + \ln x) \frac{\partial f}{\partial y} + \eta'(x,y,y') \frac{\partial f}{\partial y'} + \eta''(x,y,y',y'') \frac{\partial f}{\partial y''} = 0, \qquad (2\text{-}62)$$

in which

$$\eta'(x,y,y') = - \frac{2}{x} - (1 + \ln x) y', \qquad (2\text{-}63)$$

and

$$\eta''(x,y,y',y'') = \frac{2}{x^2} - \frac{y'}{x} - 2 y'' (1 + \ln x). \qquad (2\text{-}64)$$

The first-order system equivalent to Eq. (2-62) is

$$\frac{dx}{x \ln x} = \frac{dy}{-2(1+\ln x)} = \frac{dy'}{\eta'(x,y,y')}$$

$$= \frac{dy''}{\eta''(x,y,y',y'')} . \qquad (2\text{-}65)$$

From the first and second members, we have

$$\frac{dx}{x \ln x} = \frac{dy}{-2(1+\ln x)} , \qquad (2\text{-}66)$$

and the solution of this gives a group invariant in the form

$$u(x,y) = y + 2 \ln (x \ln x). \qquad (2\text{-}67)$$

The first and third members of Eq. (2-65) give

$$\frac{dx}{x \ln x} = \frac{-dy'}{\frac{2}{x} + (1+\ln x)\, y'} , \qquad (2\text{-}68)$$

the solution of which produces the first differential invariant

$$u'(x,y,y') = (xy' + 2) \ln x. \qquad (2\text{-}69)$$

The first and fourth members of Eq. (2-65) yield

$$\frac{dx}{x \ln x} = \frac{dy''}{\frac{2}{x^2} - \frac{y'}{x} - 2(1+\ln x)\, y''} , \qquad (2\text{-}70)$$

and the second differential invariant that comes out of this relation is

$$u''(x,y,y',y'') = (\ln x)^2 (x^2 y'' + xy'). \qquad (2\text{-}71)$$

Since Eq. (2-60) is an arbitrary function of the group invariant and first and second differential invariants just obtained, the proposition is established.

Equation (2-55) is obtained as a special case of Eq. (2-60) when the arbitrary function in Eq. (2-60) is taken so that

$$u''(x,y,y',y'') + s \exp[u(x,y)] = 0. \qquad (2\text{-}72)$$

Equation (2-55) also admits a two-parameter group.

Proposition 2.10. Equation (2-55) is invariant under the two-parameter group of point transformations with the basis transformations

$$U_1 f = x \frac{\partial f}{\partial x} - 2 \frac{\partial f}{\partial y} \qquad (2\text{-}73)$$

and

$$U_2 f = x \ln x \frac{\partial f}{\partial x} - 2\,(1+\ln x) \frac{\partial f}{\partial y} . \qquad (2\text{-}74)$$

Proof: The invariance of Eq. (2-55) under each of the groups of the basis transformations has already been shown above. Since the commutator of the basis operators assumes the form $(U_1 U_2) f = U_1 f$, and also since $U_2 f \neq \rho(x,y)\, U_1 f$, the above basis transformations generate a two-parameter Lie group of the third type.

Equation (2-55) is a special case of Eq. (2-60). It is also a special case of the general form of a second-order differential equation that admits the two-parameter group generated by the basis transformations of Eqs. (2-73) and (2-74).

Proposition 2.11. The second-order, ordinary differential equation

$$\Phi \left\{ y + \ln\left[\frac{x^2}{(2+xy')^2}\right], \frac{x^2 y'' + xy'}{(2+xy')^2} \right\} = 0, \qquad (2\text{-}75)$$

in which Φ is an arbitrary function of the two indicated arguments, comprises the general form of such an equation that is invariant under the two-parameter group of point transformations generated by the basis transformations

$$U_1 f = x \frac{\partial f}{\partial x} - 2 \frac{\partial f}{\partial y} \qquad (2\text{-}76)$$

and

$$U_2 f = x \ln x \frac{\partial f}{\partial x} - 2\,(1+\ln x) \frac{\partial f}{\partial y} . \qquad (2\text{-}77)$$

Proof: This proposition is a direct consequence of the fact that the two operators of the second extensions of the two basis transformations of Eqs. (2-76) and (2-77) annihilate the relation of Eq. (2-75).

As an example of Eq. (2-75), in which a specific form is chosen for the arbitrary function, Φ, we may take

$$\frac{x^2 y'' + xy'}{(2+xy')^2} + s \exp\left\{ y + \ln\left[\frac{x^2}{(2+xy')^2}\right]\right\} = 0, \qquad (2\text{-}78)$$

which simplifies down to the Poisson-Boltzmann equation in infinite cylindrical geometry.

2.8 General Integrals of the Poisson-Boltzmann Equation in Infinite Cylindrical Geometry

Proposition 2.10 shows that the Poisson-Boltzmann equation in infinite cylindrical geometry admits the two-parameter Lie group with the basis transformations of Eqs. (2-73) and (2-74). This fact may be exploited to effect the integration of Eq. (2-55) in closed form.

The canonical variables of the two-parameter group generated by the basis transformations contained in Eqs. (2-73) and (2-74) are

$$X = x^{-1} \exp(-y/2) \qquad (2\text{-}79)$$

and

$$Y = \ln x + x^{-1} \exp(-y/2). \qquad (2\text{-}80)$$

The introduction of these last two relations into Eq. (2-55) produces its canonical form

$$X \frac{d^2Y}{dX^2} = \left(1 - \frac{dY}{dX}\right)\left[1 + \frac{s}{2}\left(1 - \frac{dY}{dX}\right)^2\right]. \qquad (2\text{-}81)$$

If we let

$$u = 1 - \frac{dY}{dX} \qquad (2\text{-}82)$$

in Eq. (2-81), it becomes

$$\frac{du}{u\left(u^2 + \frac{2}{s}\right)} = - \frac{s}{2}\frac{dX}{X}. \qquad (2\text{-}83)$$

Consequently, a first integral of Eq. (2-81) is

$$1 - \frac{dY}{dX} = \pm\sqrt{\frac{2C_1}{s}}\frac{1}{\sqrt{X^2 - C_1}}, \qquad (2\text{-}84)$$

in which C_1 is an arbitrary constant. A second quadrature yields the general solution of Eq. (2-81),

$$Y = C_2 + X \mp \sqrt{\frac{2C_1}{s}} \ln\left[X + \sqrt{X^2 - C_1}\right], \qquad (2\text{-}85)$$

in which C_2 is the second arbitrary constant. Substituting Eqs. (2-79) and (2-80) into Eq. (2-85) produces the relation

$$\ln x = C_2 \mp \sqrt{\frac{2C_1}{s}} \ln\left[\frac{\exp(-y/2)}{x} + \sqrt{\frac{\exp(-y)}{x^2} - C_1}\right], \qquad (2\text{-}86)$$

which is the general solution of Eq. (2-55), the Poisson-Boltzmann equation in infinite cylindrical geometry. The arbitrary constants and choice of sign implied in Eq. (2-86) depend upon the boundary conditions.

If the homogeneous Dirichlet boundary condition, $y(1) = 0$, is imposed, we take the plus sign in the second term on the right-hand side of Eq. (2-86) and find that

$$C_2 = - \sqrt{\frac{2C_1}{s}} \ln\left(1 + \sqrt{1 - C_1}\right). \qquad (2\text{-}87)$$

It follows that

$$\ln x = \sqrt{\frac{2C_1}{s}} \ln\left[\frac{x^{-1}\exp(-y/2) + \sqrt{x^{-2}\exp(-y) - C_1}}{1 + \sqrt{1 - C_1}}\right], \qquad (2\text{-}88)$$

which can be written in the alternative form,

$$\ln x = \sqrt{\frac{2C_1}{s}}\left\{\operatorname{arccosh}\left[\frac{x^{-1}\exp(-y/2)}{\sqrt{C_1}}\right] - \operatorname{arccosh}\frac{1}{\sqrt{C_1}}\right\}. \qquad (2\text{-}89)$$

The resolution of Eq. (2-89) for the dependent variable, y, produces, first of all, the quadratic equation,

$$X^2 - 2\cosh\left(\sqrt{\frac{s}{2C_1}}\ln x\right)X + 1 + C_1 \sinh^2\left(\sqrt{\frac{s}{2C_1}}\ln x\right) = 0, \qquad (2\text{-}90)$$

and, finally,

$$\exp(-y/2) = x\cosh\left(\sqrt{\frac{s}{2C_1}}\ln x\right) \pm x\sqrt{1 - C_1}\sinh\left(\sqrt{\frac{s}{2C_1}}\ln x\right), \qquad (2\text{-}91)$$

which comprises the <u>two</u> solutions of Eq. (2-90). If we impose the homogeneous Neumann boundary condition, $y'(0) = 0$, it follows from Eq. (2-91) that we must take $C_1 = s/2$, so that

$$\exp(-y/2) = x\cosh(\ln x) \pm x\sqrt{1 - s/2}\sinh(\ln x). \qquad (2\text{-}92)$$

This result simplifies to the form

$$y = 2\ln\left[\frac{2}{(1 \pm \sqrt{1 - s/2})\,x^2 + 1 \mp \sqrt{1 - s/2}}\right]. \qquad (2\text{-}93)$$

It is a consequence of the relation of Eq. (2-93) that the inequality,

$$s \le s_{max} = 2, \qquad (2\text{-}94)$$

must be satisfied if Eq. (2-93) is to predict real values for the solution of the Poisson-Boltzmann equation in infinite cylindrical geometry. If $s = s_{max}$, then Eq. (2-93) becomes

$$y = 2\ln\left(\frac{2}{1 + x^2}\right), \qquad (2\text{-}95)$$

which has the maximum value

$$y_{max} = \ln 4, \qquad (2\text{-}96)$$

at $x = 0$. If $s < s_{max}$, the two solutions of Eq. (2-93) are

$$y_1 = \ln 4 - 2\ln\left[1 - \sqrt{1 - s/2} + \left(1 + \sqrt{1 - s/2}\right)x^2\right] \qquad (2\text{-}97)$$

and

$$y_2 = \ln 4 - 2\ln\left[1 + \sqrt{1 - s/2} + \left(1 - \sqrt{1 - s/2}\right)x^2\right]. \qquad (2\text{-}98)$$

The maximum value of the solution in Eq. (2-97) at $x = 0$ is

$$y_{1,max} = \ln 4 - 2\ln(1 - \sqrt{1 - s/2}), \qquad (2\text{-}99)$$

and that of Eq. (2-98) is

$$y_{2,max} = \ln 4 - 2 \ln (1 + \sqrt{1 - s/2}). \qquad (2\text{-}100)$$

Note that $y_{1,max} > y_{max}$ even though $s < s_{max}$. A higher center temperature for a smaller source term is predicted by the solution of Eq. (2-97), which may not, therefore, be a stable and observable solution. However, it is noted that $y_{2,max} < y_{max}$ for the solution of Eq. (2-98).

The preceding discussion of the Poisson-Boltzmann equation in infinite cylindrical geometry may be compared with that given by Chambre,[2] who also obtained the results contained in Eqs. (2-94) through (2-96), but from a rather different point of view. The general solution of Eq. (2-55) as obtained in Eq. (2-86) does not appear to have been established previously.

3. OCCURRENCE OF LIE GROUPS IN THE ANALYSIS OF SPHERICALLY SYMMETRIC, IRROTATIONAL FLOWS OF AN INVISCID, INCOMPRESSIBLE FLUID

3.1 Introduction

The cognate problems of analyzing the collapse of a spherical bubble and the expansion of a spherical cavity in an infinite expanse of fluid have been treated by Rayleigh[3] and Lamb.[4] The invariance properties of certain nonlinear, ordinary differential equations under groups of point transformations are established in this section for these problems.

3.2 Formulation of the Problem

The spherical cavity collapse and expansion problems considered by Rayleigh[3] and Lamb[4] can be regarded as special cases of a common formulation to display the underlying physical limitations inherent in the discussion.

If we let $\rho \equiv$ density, $\vec{V} \equiv$ velocity field, $p \equiv$ pressure, and $\vec{\Phi} \equiv$ body force per unit mass, then the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \vec{V}\cdot \text{grad}\, \rho + \rho \,\text{div}\, \vec{V} = 0, \qquad (3\text{-}1)$$

and the equation of motion for an inviscid fluid is

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\text{grad})\,\vec{V}\right] = \rho\,\vec{\Phi} - \text{grad}\, p. \qquad (3\text{-}2)$$

If the body force is derivable from a potential, Ω, so that

$$\vec{\Phi} = -\,\text{grad}\,\Omega, \qquad (3\text{-}3)$$

and if it is noted that

$$\text{grad}\int\frac{dp}{\rho} = \frac{1}{\rho}\,\text{grad}\, p, \qquad (3\text{-}4)$$

the equation of motion becomes

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\text{grad})\,\vec{V} = -\,\text{grad}\left(\Omega + \int\frac{dp}{\rho}\right). \qquad (3\text{-}5)$$

With the vector identity

$$(\vec{V}\cdot\text{grad})\,\vec{V} = \text{grad}\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) - \vec{V} \times \text{curl}\,\vec{V}, \qquad (3\text{-}6)$$

Eq. (3-5) reduces to

$$\frac{\partial \vec{V}}{\partial t} - \vec{V} \times \text{curl}\,\vec{V} = -\,\text{grad}\left(\Omega + \frac{\vec{V}\cdot\vec{V}}{2} + \int\frac{dp}{\rho}\right). \qquad (3\text{-}7)$$

The assumption of an irrotational flow, that is, $\text{curl}\,\vec{V} = 0$, so that $\vec{V} = -\,\text{grad}\, P$, wherein $P \equiv$ velocity potential, produces from Eq. (3-7) the form

$$\frac{\partial}{\partial t}\,\text{grad}\, P = \text{grad}\left(\Omega + \frac{\vec{V}\cdot\vec{V}}{2} + \int\frac{dp}{\rho}\right). \qquad (3\text{-}8)$$

Integration of this last relation produces

$$C(t) = -\frac{\partial P}{\partial t} + \Omega + \frac{\vec{V}\cdot\vec{V}}{2} + \int\frac{dp}{\rho}, \qquad (3\text{-}9)$$

in which $C(t)$ is an arbitrary function of the time.

We now apply Eqs. (3-1) and (3-9) to spherically symmetric flows around a spherical cavity surrounded by an infinite extent of a fluid already assumed to be inviscid. If the further assumption is made that the fluid is also incompressible, the continuity equation is simply $\text{div}\,\vec{V} = 0$, and, consequently, the velocity potential satisfies Laplace's equation, $\text{div grad}\, P = 0$, which for a spherically symmetric flow is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dP}{dr}\right) = 0. \qquad (3\text{-}10)$$

This integrates to

$$P = -\frac{c_1}{r} + c_2, \qquad (3\text{-}11)$$

in which we take $c_2 = 0$ for a vanishing velocity potential at infinity. Let $\dot{R} \equiv$ speed of the surface between the spherical cavity and the surrounding fluid. Then

$$\dot{R} = V_r(R) = -\left.\frac{\partial P}{\partial r}\right|_{r=R} = -\frac{c_1}{R^2}. \qquad (3\text{-}12)$$

With the value of the constant c_1 as given by Eq. (3-12), the velocity potential is

$$P = \frac{R^2\dot{R}}{r}. \qquad (3\text{-}13)$$

Now assume that body forces are negligible, so $\Omega = 0$, and evaluate the time dependent Bernoulli equation, Eq. (3-9), at infinity to get

$$C(t) = \frac{p_\infty}{\rho}, \qquad (3\text{-}14)$$

wherein p_∞ = fluid pressure at infinity. At any radial position, r, we have

$$\frac{p_\infty}{\rho} = -\frac{\partial P}{\partial t} + \frac{1}{2} V_r^2 + \frac{p}{\rho} . \qquad (3\text{-}15)$$

Because

$$V_r = -\frac{\partial P}{\partial r} = \frac{R^2\dot{R}}{r^2} , \qquad (3\text{-}16)$$

Eq. (3-15) becomes

$$\frac{p_\infty}{\rho} = -\frac{\partial}{\partial t}\left(\frac{R^2\dot{R}}{r}\right) + \frac{1}{2}\frac{R^4\dot{R}^2}{r^4} + \frac{p}{\rho} , \qquad (3\text{-}17)$$

which in turn simplifies to

$$\frac{1}{r}\left(R^2\ddot{R} + 2R\,\dot{R}^2\right) - \frac{R^4\dot{R}^2}{2r^4} = \frac{1}{\rho}\,(p - p_\infty). \qquad (3\text{-}18)$$

Evaluation of Eq. (3-18) at the cavity-fluid interface yields the relation

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho}\left[p(R) - p_\infty\right] \qquad (3\text{-}19)$$

as the nonlinear differential equation that governs the time dependence of the radius of the cavity. In this equation, $p(R)$ is interpreted as the pressure of the gas in the cavity.

If we assume that $p_\infty >>> p(R)$ for all values of the cavity radius, R, then Eq. (3-19) becomes

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = -\frac{p_\infty}{\rho} . \qquad (3\text{-}20)$$

This equation underlies Rayleigh's discussion[3] of the bubble collapse problem.

However, if we assume that $p(R) >>> p_\infty$ for all values of the cavity radius, then Eq. (3-19) is approximated by

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{p(R)}{\rho} . \qquad (3\text{-}21)$$

When the gas expansion is adiabatic,

$$p(R) = p_{go}\left(\frac{R_o}{R}\right)^{3\gamma} \qquad (3\text{-}22)$$

in which $\gamma \equiv$ specific heat ratio of the gas and p_{go} is the initial gas pressure that corresponds to the initial cavity radius, R_o. Combining Eqs. (3-21) and (3-22) produces

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{p_{go}}{\rho}\left(\frac{R_o}{R}\right)^{3\gamma} . \qquad (3\text{-}23)$$

This nonlinear differential equation is the basis of Lamb's discussion[4] of the early stages of a submarine explosion. The physical limitations inherent in its formulation have been delineated above.

Lamb[4] was of the opinion that Eq. (3-23) is integrable by quadratures only in the case for which $\gamma = 4/3$. We now establish the fact that Eq. (3-23) is invariant under certain groups of point transformations and to exploit this fact to obtain further closed form, analytic solutions of Eq. (3-23) by quadratures.

3.3 Lie Groups Applicable to the Submarine Explosion Problem

In Eq. (3-23), let $y = R$, $t = x$, and $K = p_{go}R_o^{3\gamma}/\rho$, so that we may write

$$yy'' + \frac{3}{2}(y')^2 = K\,y^{-3\gamma}. \qquad (3\text{-}24)$$

This nonlinear differential equation is invariant under the group of translations parallel to the x-axis because it is autonomous. We also find the following result.

Proposition 3.1. The one-parameter group of point transformations generated by the infinitesimal transformation represented by the symbol

$$Uf = x\frac{\partial f}{\partial x} + \frac{1}{\alpha}\,y\frac{\partial f}{\partial y} , \qquad (3\text{-}25)$$

in which

$$\alpha \equiv 1 + \frac{3\gamma}{2} , \qquad (3\text{-}26)$$

is admitted by Eq. (3-24).

Proof: The linear, first-order partial differential equation equivalent to Eq. (3-24) is

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \left[K\,y^{-3\gamma-1} - \frac{3}{2}y^{-1}(y')^2\right]\frac{\partial f}{\partial y'} = 0. \qquad (3\text{-}27)$$

The symbol of the once-extended group generated by Eq. (3-25) is

$$U'f = x\frac{\partial f}{\partial x} + \frac{y}{\alpha}\frac{\partial f}{\partial y} + \left(\frac{1}{\alpha} - 1\right)y'\frac{\partial f}{\partial y'} . \qquad (3\text{-}28)$$

The value of the commutator that comes out of the symbols of Eqs. (3-27) and (3-28) is

$$(U'A)f = -Af, \qquad (3\text{-}29)$$

which establishes the proposition.

A further result follows.

Proposition 3.2. The nonlinear, second-order differential equation,

$$yy'' + \frac{3}{2}(y')^2 = K\,y^{-3\gamma}, \qquad (3\text{-}30)$$

is invariant under the two-parameter group of point transformations whose basis transformations are represented by the symbols

$$U_1f = \frac{\partial f}{\partial x} \qquad (3\text{-}31)$$

and

$$U_2f = x\frac{\partial f}{\partial x} + \frac{y}{\alpha}\frac{\partial f}{\partial y} . \qquad (3\text{-}32)$$

Proof: The symbols of Eqs. (3-31) and (3-32) generate a two-parameter group because they are independent and because their commutator assumes the form $(U_1U_2)f = U_1f$. The invariance of Eq. (3-30) under the transformations generated by Eqs. (3-31) and (3-32) has already been established.

The two-parameter group generated by the basis transformations included in Eqs. (3-31) and (3-32) is of the third type in the sense of Lie's definition because they are unconnected and because their commutator takes the form previously indicated.

3.4 Reduction to Canonical Form and Integration of the Cavity-Radius Differential Equation

Because of the invariance property of Eq. (3-30) established in Proposition 3.2, this differential equation may be reduced to its canonical form which is integrable by quadratures. The appropriate canonical variables are

$$Y = x \tag{3-33}$$

and

$$X = y^\alpha . \tag{3-34}$$

In terms of these canonical variables we have

$$y' = \frac{X^{\frac{1-\alpha}{\alpha}}}{\alpha} \left(\frac{dY}{dX}\right)^{-1} \tag{3-35}$$

and

$$yy'' = \frac{1-\alpha}{\alpha^2} X^{\frac{2}{\alpha}(1-\alpha)} \left(\frac{dY}{dX}\right)^{-2} - \frac{X}{\alpha}\left(\frac{dY}{dX}\right)^{-3} \frac{d^2Y}{dX^2} X^{\frac{2}{\alpha}(1-\alpha)} . \tag{3-36}$$

The substitution of Eqs. (3-34) through (3-36) into Eq. (3-30) produces

$$X \frac{d^2Y}{dX^2} = \frac{1}{\alpha}\left(\frac{5}{2} - \alpha\right) \frac{dY}{dX} - \alpha K \left(\frac{dY}{dX}\right)^3 , \tag{3-37}$$

which is the canonical form of Eq. (3-30).

Since the solution of Eq. (3-30) is to satisfy the initial conditions, $y(0) = y_o$ and $y'(0) = 0$, it follows that Eq. (3-37) is to be solved subject to the conditions

$$Y = 0 \text{ when } X = y_o^\alpha \tag{3-38}$$

and

$$\frac{dY}{dX} \to \infty \text{ when } X = y_o^\alpha . \tag{3-39}$$

In Eq. (3-37), let

$$u = \frac{dY}{dX} \tag{3-40}$$

and

$$\Gamma^2 = \frac{\alpha - 5/2}{\alpha^2 K} ; \tag{3-41}$$

it then becomes

$$\int_u^\infty \frac{du}{u\,(u^2 + \Gamma^2)} = -\alpha K \int_X^{y_o^\alpha} \frac{dX}{X} , \tag{3-42}$$

which integrates out to provide

$$\frac{dY}{dX} = \frac{\Gamma}{\sqrt{\left(\frac{X}{y_o^\alpha}\right)^{\frac{3(\gamma-1)}{\alpha}} - 1}} . \tag{3-43}$$

It follows that

$$Y = \Gamma \int_{y_o^\alpha}^{X} \frac{dX}{\sqrt{\left(\frac{X}{y_o^\alpha}\right)^{\frac{3(\gamma-1)}{\alpha}} - 1}} , \tag{3-44}$$

and reverting to the original variables produces

$$x = \Gamma \int_{y_o^\alpha}^{y^\alpha} \frac{dX}{\sqrt{\left(\frac{X}{y_o^\alpha}\right)^{\frac{3(\gamma-1)}{\alpha}} - 1}} . \tag{3-45}$$

This last relation may also be written as

$$x = \Gamma\, y_o^\alpha \int_1^{\left(\frac{y}{y_o}\right)^\alpha} \frac{d\tau}{\sqrt{\tau^{\frac{3}{\alpha}(\gamma-1)} - 1}} . \tag{3-46}$$

The preceding discussion has established the following result.

Proposition 3.3. The solution of the nonlinear differential equation

$$yy'' + \frac{3}{2}(y')^2 = K\, y^{-3\gamma} , \tag{3-47}$$

subject to the two initial conditions, $y(0) = y_o$ and $y'(0) = 0$, is given by the integral representation,

$$x = \Gamma\, y_o^\alpha \int_1^{\left(\frac{y}{y_o}\right)^\alpha} \frac{d\tau}{\sqrt{\tau^{\frac{3}{\alpha}(\gamma-1)} - 1}} , \tag{3-48}$$

wherein

$$\alpha = 1 + \frac{3\gamma}{2} \tag{3-49}$$

and

$$\Gamma = \sqrt{\frac{3(\gamma-1)}{2\,K\left(1 + \frac{3\gamma}{2}\right)^2}}\,. \tag{3-50}$$

A solution of Eq. (3-47) was obtained by Lamb[4] for the special case in which the specific heat ratio is assigned the value $\gamma = 4/3$. This solution also comes out of Eq. (3-48), that reduces to

$$x = \Gamma\, y_o^{\alpha} \int_1^{\left(\frac{y}{y_o}\right)^3} \frac{d\tau}{\sqrt{\tau^{1/3} - 1}} \tag{3-51}$$

when $\gamma = 4/3$. Because in this case

$$\Gamma = \frac{1}{3\sqrt{2K}}\,, \tag{3-52}$$

and

$$\int_1^{\left(\frac{y}{y_o}\right)^3} \frac{d\tau}{\sqrt{\tau^{1/3} - 1}} = \frac{2}{5}\sqrt{\frac{y}{y_o} - 1}\left[3\left(\frac{y}{y_o}\right)^2 + 4\left(\frac{y}{y_o}\right) + 8\right]\,, \tag{3-53}$$

it follows that Eq. (3-51) simplifies to

$$x = \frac{2y_o^3}{15\sqrt{2K}}\sqrt{\frac{y}{y_o} - 1}\left[3\left(\frac{y}{y_o}\right)^2 + 4\left(\frac{y}{y_o}\right) + 8\right]\,, \tag{3-54}$$

which is equivalent to that obtained by Lamb[4] by another method. However, the integral representation of Eq. (3-48) also provides further analytic solutions of Eq. (3-47) for additional values of the specific heat ratio.

For example, let

$$\frac{1}{q} = \frac{3(\gamma-1)}{\alpha} \tag{3-55}$$

in Eq. (3-48). Then with $\tau = t^{2q}$, it becomes

$$x = \Gamma\, y_o^{\alpha}\, 2q \int_1^{\left(\frac{y}{y_o}\right)^{\frac{\alpha}{2q}}} \frac{dt\; t^{2q-1}}{\sqrt{t^2 - 1}}\,. \tag{3-56}$$

If we set $q = 3$ in this last relation, it can be reduced to the result already given in Eq. (3-54). Further reductions follow.

(1) If $q = 5/2$, we have $\gamma = 17/12$ and

$$x = \frac{y_o^{25/8}}{\sqrt{40K}}\left\{\left[2\left(\frac{y}{y_o}\right)^{15/8} + 3\left(\frac{y}{y_o}\right)^{5/8}\right]\sqrt{\left(\frac{y}{y_o}\right)^{5/4} - 1} + 3\ln\left[\left(\frac{y}{y_o}\right)^{5/8} + \sqrt{\left(\frac{y}{y_o}\right)^{5/4} - 1}\right]\right\}. \tag{3-57}$$

(2) If $q = 2$, we have $\gamma = 14/9$ and

$$x = \sqrt{\frac{6}{5K}}\, y_o^{10/3}\left\{\frac{1}{3}\left[\left(\frac{y}{y_o}\right)^{5/3} - 1\right]^{3/2} + \sqrt{\left(\frac{y}{y_o}\right)^{5/3} - 1}\right\}. \tag{3-58}$$

(3) If $q = 3/2$, we have $\gamma = 11/6$ and

$$x = \frac{y_o^{15/4}}{\sqrt{5K}}\left\{\left(\frac{y}{y_o}\right)^{5/4}\sqrt{\left(\frac{y}{y_o}\right)^{5/2} - 1} + \ln\left[\left(\frac{y}{y_o}\right)^{5/4} + \sqrt{\left(\frac{y}{y_o}\right)^{5/2} - 1}\right]\right\}. \tag{3-59}$$

Equations (3-57) through (3-59) are the analytic solutions of Eq. (3-47) that are valid for the indicated values of the specific heat ratio, γ.

Further integrals of Eq. (3-47) may be constructed from Eq. (3-48).

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